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A CHARACTERIZATION OF STRONG MEASURE ZERO SETS

BY

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ABSTRACT

We show that a set $X \subseteq \mathbf{R}$ has strong measure zero iff for every closed measure zero set $F \subseteq \mathbf{R}$, $F + X$ has measure zero.

A set $X \subseteq \mathbf{R}$ is strongly null (has strong measure zero) iff for every sequence of positive real numbers $\langle \epsilon_n : n \in \omega \rangle$ there exists a sequence $\langle X_n : n \in \omega \rangle$ of subsets of $\mathbf{R}$ such that $X \subseteq \bigcup_n X_n$ and the diameter of $X_n < \epsilon_n$ (see [Bo], [Mi]). Galvin, Mycielski and Solovay [GMS] characterized strongly null sets in the following way (see [G] or [Mi] for a proof):

THEOREM (Galvin, Mycielski and Solovay): *Let $X \subseteq \mathbf{R}$. The following conditions are equivalent:*

(a) *X is strongly null;*
(b) *every dense $\mathbf{G}_\delta$ set contains a translate of X;*
(c) *every open dense set contains a translate of X.*

The Galvin–Mycielski–Solovay theorem can be rephrased as follows. A set $X \subseteq \mathbf{R}$ is strongly null iff $D + X \neq \mathbf{R}$ for every meager (equivalently, nowheredense) set D. Indeed, just note that for $t \in \mathbf{R}$, $t \notin D + X$ iff $(-t) + X \subseteq \mathbf{R} \setminus (-D)$.

We shall prove the following theorem.

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THEOREM: *A set $X \subseteq \mathbf{R}$ is strongly null iff $F + X$ is null for every closed null set $F \subseteq \mathbf{R}$.*

Strongly null sets can be defined for any metric space. For technical reasons we choose to work with the Cantor set ${}^{\omega}2$ rather than $\mathbf{R}$. It is not hard to transfer our arguments to $\mathbf{R}$ and $\mathbf{R}/\mathbf{Z}$ (and then, in fact, to any abelian locally compact finite dimensional Polish group).

Notation: ω is the set of nonnegative integers. For $n \in \omega$, $n = \{0, 1, \ldots, n-1\}$. For $A \subseteq \omega$ (finite or infinite), we endow ${}^{A}2$ with the product structure arising from considering $2 = \{0, 1\}$ as the group of addition mod 2 with the discrete metric and measure μ such that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. We denote the product measure in ${}^{A}2$ by μ_A and usually drop the subscript. We say that sets $T_i \subseteq {}^{A}2$ $(i \in I)$ are measure independent if $\mu(\bigcap_{i \in J} T_i) = \prod_{i \in J} \mu(T_i)$ for every finite $J \subseteq I$. Note that if T_i's are independent, then so are $({}^{A}2 \setminus T_i)$'s

For $\tau \in {}^{A}2$, let $[\tau] = \{t \in {}^{\omega}2 \colon \tau \subseteq t\}$. For $T \subseteq {}^{A}2$, let $[T] = \bigcup_{\tau \in T}[\tau]$. Given sets S_n $(n \in \omega)$, we write $\bigvee_n S_n$ for the upper limit $\bigcap_m \bigcup_{n>m} S_n$ and $\bigwedge_n S_n$ for the lower limit $\bigcup_m \bigcap_{n>m} S_n$. We also write $\exists^{\infty} n$ for $\forall m \exists n \geq m$ and $\forall^{\infty} n$ for $\exists m \forall n \geq m$. Thus, $t \in \bigvee_n [\tau_n]$ iff $\exists^{\infty} n$ $\tau_n \subseteq t$ and $t \in \bigwedge_n [\tau_n]$ iff $\forall^{\infty} n$ $\tau_n \subseteq t$.

If $B \subseteq A \subseteq \omega$, we identify ${}^{A}2$ with ${}^{B}2 \times {}^{A \setminus B}2$. For $\tau \in {}^{B}2$ and $G \subseteq {}^{A}2$, let $G_\tau = \{\sigma \in {}^{A \setminus B}2 \colon \tau \cup \sigma \in G\}$ (we view G_τ as the section of $G \subseteq {}^{B}2 \times {}^{A \setminus B}2$ determined by τ).

Let ${}^{<\omega}2 = \bigcup_n {}^{n}2$.

It is not hard to see that a set $X \subseteq {}^{\omega}2$ is strongly null iff for every sequence $\langle a_n \colon n \in \omega \rangle$ of integers there exist $\sigma_n \in {}^{a_n}2$ such that $X \subseteq \bigcup_n [\sigma_n]$. We can equally well write here $X \subseteq \bigvee_n [\sigma_n]$ (split ω into infinitely many infinite sets).

The following lemma, however trivial, is the key to everything.

LEMMA 0: *Let $m \geq n + 2^n k$, $k, n, m \in \omega$. There exists $T \subseteq {}^{m}2$ such that $\mu(T) = 2^{-k}$ and for any $\langle \sigma_i, \tau_i \rangle \in {}^{n}2 \times {}^{[n,m)}2$ $(i \in I)$ with σ_i's distinct, the sets $T + \langle \sigma_i, \tau_i \rangle$ $(i \in I)$ are measure independent.*

Proof: Find 2^n disjoint sets $u_\sigma \subseteq [n, m)$ $(\sigma \in {}^{n}2)$, each of size k. Let $T_\sigma = \{\tau \in {}^{[n,m)}2 \colon \tau | u_\sigma \equiv 0\}$. Then $\mu(T_\sigma) = 2^{-k}$. Also, if $\sigma_i \in {}^{n}2$ $(i \in I)$ are distinct and $\tau_i \in {}^{[n,m)}2$ $(i \in I)$ are arbitrary, then $T_{\sigma_i} + \tau_i$ $(i \in I)$ are measure independent.

Let $T = \bigcup_{\sigma \in {}^{n}2} \{\sigma\} \times T_\sigma$. Then $\mu(T) = 2^{-k}$. If now $\sigma_i \in {}^{n}2$ $(i \in I)$ are distinct,

$\tau_i \in {}^{[n,m)}2$ $(i \in I)$ are arbitrary, and $J \subseteq I$, then

$$\begin{aligned}\bigcap_{i\in J} T + \langle\sigma_i, \tau_i\rangle &= \bigcap_{i\in J}\bigcup_{\sigma\in{}^n2} \{\sigma+\sigma_i\} \times (T_\sigma + \tau_i)\\ &= \bigcap_{i\in J}\bigcup_{\sigma\in{}^n2} \{\sigma\} \times (T_{\sigma+\sigma_i} + \tau_i)\\ &= \bigcup_{\sigma\in{}^n2} \{\sigma\} \times \bigcap_{i\in J}(T_{\sigma+\sigma_i} + \tau_i).\end{aligned}$$

Since for every σ, $(T_{\sigma+\sigma_i} + \tau_i)$'s are measure independent (because σ_i's are distinct), it follows that $(T + \langle\sigma_i, \tau_i\rangle)$'s are measure independent. ∎

Now comes the basic lemma.

LEMMA 1: *For every comeager set $H \subseteq {}^\omega 2$ there is a closed null set $F \subseteq {}^\omega 2$ such that for every $X \subseteq {}^\omega 2$ with $F + X$ null there is $t \in {}^\omega 2$ with $t + X \subseteq H$.*

Proof: Fix a comeager set $H \subseteq {}^\omega 2$. Choose integers

$$\begin{aligned} a_0 &= 0,\\ a_n &= a_n^0 < a_n^1 < \cdots < a_n^{n\cdot 2^{a_n}} = b_n,\\ a_{n+1} &= b_n + 2^{b_n - a_n},\end{aligned}$$

and sequences

$$\sigma_n^i\colon [a_n^i, a_n^{i+1}) \to 2,$$

such that

$$\{s \in {}^\omega 2\colon \exists^\infty n\ \exists i\ \sigma_n^i \subseteq s\} \subseteq H.$$

(If $H \supseteq \bigcap_n H_n$, H_n open dense, choose σ_n^i so that $[\sigma_n^i] \subseteq \bigcap_{m\le n} H_m$.)

Find $F_n \subseteq {}^{[a_n,a_{n+1})}2$ with $\mu(F_n) = 1/2$ such that, whenever $\sigma_i \in {}^{[a_n,a_{n+1})}2$ $(i \in I)$ have distinct restrictions to $[a_n, b_n)$, then $(F_n + \sigma_i)$'s are measure independent. Let $F = \bigcap_n [F_n]$. Then F is closed and null. Suppose that $F + X$ is null.

CLAIM: *There exist $K_n \subseteq {}^{[a_n,b_n)}2$ $(n \in \omega)$ with $|K_n| \le n \cdot 2^{a_n}$ such that*

$$\forall x \in X\ \exists^\infty n\ x|[a_n, b_n) \in K_n.$$

Proof: Let $G \subseteq {}^\omega 2$ be an open set covering ${}^{<\omega}2 + F + X$ such that $\mu(G) < \prod_n \epsilon_n$, where $\epsilon_n = 1 - 2^{-(n+1)}$. Let

$$K_n = \{\sigma|[a_n, b_n)\colon \sigma \in {}^{[a_n,a_{n+1})}2\ \&\ \exists\ \tau \in {}^{a_n}2\ F_n + \sigma \subseteq L_\tau\},$$

where

$$L_\tau = \{\sigma \in {}^{[a_n,a_{n+1})}2 \colon \mu(G_{\tau\cup\sigma}) > \mu(G_\tau)/\epsilon_n\}.$$

We have

$$|K_n| \leq n \cdot 2^{a_n}.$$

Indeed, fix $\tau \in {}^{a_n}2$. By the Fubini theorem applied to $G_\tau \subseteq {}^{[a_n,a_{n+1})}2 \times {}^{[a_{n+1},\omega)}2$, $\mu(L_\tau) < \epsilon_n$. Let $\sigma_k \in {}^{[a_n,a_{n+1})}2$ $(k < k_\tau)$ be such that $F_n + \sigma_k \subseteq L_\tau$ and all $\sigma_k | [a_n, b_n)$ are distinct. Then

$$\bigcap_k {}^{[a_n,a_{n+1})}2 \setminus (F_n + \sigma_k) \supseteq {}^{[a_n,a_{n+1})}2 \setminus L_\tau.$$

So, using independence,

$$\begin{aligned} 2^{-k_\tau} &= (1 - 1/2)^{k_\tau} \\ &\geq 1 - \mu(L_\tau) \\ &> 1 - \epsilon_n = 2^{-(n+1)}, \end{aligned}$$

hence $k_\tau \leq n$. The estimation for $|K_n|$ follows.

We shall now show that $\forall x \in X\ \exists^\infty n\ x|[a_n, b_n) \in K_n$. Fix $x \in X$. It is enough to show

$$\exists^\infty n\ \exists \tau \in {}^{a_n}2\ F_n + x|[a_n, a_{n+1}) \subseteq L_\tau.$$

Suppose this is not true. Then

$$\forall^\infty n\ \forall \tau \in {}^{a_n}2\ F_n + x|[a_n, a_{n+1}) \not\subseteq L_\tau.$$

So there is m such that

$$\forall n \geq m\ \forall \tau \in {}^{a_n}2\ \exists \sigma \in F_n + x|[a_n, a_{n+1})\ \mu(G_{\tau\cup\sigma}) \leq \mu(G_\tau)/\epsilon_n.$$

Note also that for every n and $\tau \in {}^{a_n}2$, by the Fubini theorem applied to $G_\tau \subseteq {}^{[a_n,a_{n+1})}2 \times {}^{[a_{n+1},\omega)}2$,

$$\exists \sigma \in {}^{[a_n,a_{n+1})}2\ \mu(G_{\tau\cup\sigma}) \leq \mu(G_\tau) \leq \mu(G_\tau)/\epsilon_n.$$

Now we can inductively define $t \in {}^\omega 2$ such that

$$\forall n \geq m, \quad t|[a_n, a_{n+1}) \in F_n + x|[a_n, a_{n+1})$$

and

$$\forall n \quad \epsilon_n \mu(G_{t|a_{n+1}}) \leq \mu(G_{t|a_n}).$$

Then $t \in {}^{<\omega}2 + F + x$, so $t \in G$. Since G is open, there is n with $\mu(G_{t|a_{n+1}}) = 1$. Then

$$\epsilon_0 \cdots \epsilon_n = \epsilon_0 \cdots \epsilon_n \mu(G_{t|a_{n+1}}) \leq \mu(G_{t|a_0}) = \mu(G),$$

which contradicts $\mu(G) < \prod_n \epsilon_n$. ∎ (Claim)

We shall now show how to get t with $t + X \subseteq H$. Let $K_n = \{\tau_n^i : i < n \cdot 2^{a_n}\}$. Let $t \in {}^{\omega}2$ be any extension of $\bigcup_{n,i} \sigma_n^i + \tau_n^i | [a_n^i, a_n^{i+1})$. By the claim, given $x \in X$, $\exists^\infty n \; \exists i \; x \supseteq \tau_n^i | [a_n^i, a_n^{i+1})$. So,

$$\exists^\infty n \; \exists i \; t + x \supseteq \sigma_n^i + \tau_n^i | [a_n^i, a_n^{i+1}) + \tau_n^i | [a_n^i, a_n^{i+1}) = \sigma_n^i.$$

It follows that $t + x \in H$. ∎

The following two lemmas are folklore.

LEMMA 2: *If for every open dense set $H \subseteq {}^{\omega}2$ there exists $t \in {}^{\omega}2$ with $t + X \subseteq H$, then X is strongly null.*

Proof: Fix an increasing sequence of integers $\langle a_n : n \in \omega \rangle$. Choose $\tau_n \in {}^{a_n}2$ so that $H = \bigcup_n [\tau_n]$ is dense. Let $t \in {}^{\omega}2$ be such that $X \subseteq H + t$. Then the sequence $\langle \tau_n + t | a_n : n \in \omega \rangle$ witnesses for $\langle a_n : n \in \omega \rangle$ that X is strongly null. ∎

LEMMA 3: *Suppose that $X \subseteq {}^{\omega}2$ is strongly null and $F \subseteq {}^{\omega}2$ is closed null. Then $F + X$ is null.*

Proof: Fix an increasing sequence of integers $\langle a_n : n \in \omega \rangle$ and sets $F_n \subseteq {}^{[a_n, a_{n+1})}2$ of measure $\leq 2^{-n}$ so that $F \subseteq \bigcap [F_n]$. Since X is strongly null, there exist $\tau_n \in {}^{[a_n, a_{n+1})}2$ such that $X \subseteq \bigvee_n [\tau_n]$. Now,

$$F + X \subseteq \bigvee_n [F_n + \tau_n].$$

Since $\mu(F_n + \tau_n) \leq 2^{-n}$, it follows that $F + X$ is null. ∎

Proof of Theorem: Suppose that $F + X$ is null for all closed null $F \subseteq {}^{\omega}2$. Then, by Lemma 1, X can be translated into any comeager set. So, by Lemma 2, X is strongly null. The other direction follows by Lemma 3. ∎

Note that we have also proved the Galvin–Mycielski–Solovay theorem.

The nontrivial implication in our theorem can be rephrased as follows. If $F+X$ is null for all closed null $F \subseteq \mathbf{R}$, then $F+X \neq \mathbf{R}$ for all meager $F \subseteq \mathbf{R}$. This is a distant analogue of the following theorem of Shelah [Sh].

THEOREM (Shelah): *If $F+X$ is null for all null $F \subseteq \mathbf{R}$, then $F+X$ is meager for all meager $F \subseteq \mathbf{R}$.*

We shall now modify Lemma 1 so that it would yield Shelah's theorem. First let us record the following elementary lemma.

LEMMA 4: *Let $A_n \subseteq \omega$ $(n \in \omega)$ be finite and pairwise disjoint. Let $\tau_n = A_n \times \{0\}$ and let $T_n \subseteq {}^{A_n}2$ $(n \in \omega)$ be nonempty. Then*

(a) *$\bigvee_n[\tau_n]$ and $\bigvee_n[T_n]$ are comeager;*

(b) *$\bigvee_n[T_n] = \bigwedge_n[T_n] + \bigvee_n[\tau_n] = \bigvee_n[T_n] + \bigwedge_n[\tau_n]$;*

(c) *if $Y \subseteq {}^\omega 2$ is such that $Y + \bigvee_n[T_n] \neq {}^\omega 2$, then $Y + \bigwedge_n[T_n]$ is meager.*

Proof: (a) and (b) are clear. For (c), note that if $t \notin Y + \bigvee_n[T_n]$, then $t \notin Y + \bigwedge_n[T_n] + \bigvee_n[\tau_n]$. It follows that $t + \bigvee_n[\tau_n]$ is disjoint with $Y + \bigwedge_n[T_n]$. ∎

PROPOSITION 1: *For every meager set $D \subseteq {}^\omega 2$ there exist an increasing sequence $\langle a_n : n \in \omega\rangle \in {}^\omega\omega$ and sets $F_n, T_n \subseteq {}^{[a_n, a_{n+1})}2$ of measure $\leq 2^{-n}$ (so $\bigvee_n[F_n]$, $\bigvee_n[T_n]$ are null, and $\bigwedge_n[F_n]$, $\bigwedge_n[T_n]$ are null $\mathbf{F}_\sigma$) such that $D + \bigvee_n[T_n] \neq {}^\omega 2$, $D + \bigwedge_n[T_n]$ is meager, and*

(♣) *every $X \subseteq {}^\omega 2$ for which $\bigwedge_n[F_n] + X$ is null can be translated into $\bigvee_n[T_n]$ (thus, if $\bigwedge_n[F_n] + X$ is null then $D + X \neq {}^\omega 2$);*

(♠) *every $X \subseteq {}^\omega 2$ for which $\bigvee_n[F_n] + X$ is null can be translated into $\bigwedge_n[T_n]$ (thus, if $\bigvee_n[F_n] + X$ is null then $D + X$ is meager).*

Proof: Fix a meager set $D \subseteq {}^\omega 2$. Let $c_n \in \omega$ be large enough with respect to n (e.g, $c_n = 2^{2n+1}$). Choose integers

$$a_0 = 0,$$
$$a_n = a_n^0 < a_n^1 < \cdots < a_n^{c_n \cdot 2^{a_n}} = b_n,$$
$$a_{n+1} = b_n + 2^{b_n - a_n} \cdot n,$$

and sequences

$$\sigma_n^i \colon [a_n^i, a_n^{i+1}) \to 2,$$

such that

$$\{s \in {}^\omega 2 \colon \exists^\infty n\ \exists i\ \sigma_n^i \subseteq s\} \subseteq {}^\omega 2 \setminus D.$$

Make sure that

$$a_n^{i+1} - a_n^i \geq d_n,$$

where $d_n \in \omega$ are such that

$$2^{-d_n} \cdot c_n \cdot 2^{a_n} \leq 2^{-n}.$$

Define now

$$T_n = \{\tau \in {}^{[a_n,a_{n+1})}2\colon \exists i\ \tau|[a_n^i, a_n^{i+1}) \equiv 0\}.$$

Then

$$\mu(T_n) \leq 2^{-d_n} \cdot c_n \cdot 2^{a_n} \leq 2^{-n}.$$

Also it is not hard to see that if $s \in {}^\omega 2$ is such that $\bigcup_{i,n} \sigma_n^i \subseteq s$, then

$$s + \bigvee_n [T_n] \subseteq {}^\omega 2 \setminus D.$$

It follows that $D + \bigvee_n [T_n] \neq {}^\omega 2$, hence, by Lemma 4, $D + \bigwedge_n [T_n]$ is meager.

Now choose sets $F_n \subseteq {}^{[a_n,a_{n+1})}2$ $(n \in \omega)$ so that $\mu(F_n) = 2^{-n}$ and whenever $\sigma_i \in {}^{[a_n,a_{n+1})}2$ $(i \in I)$ have distinct restrictions to $[a_n, b_n)$ then $(F_n + \sigma_i)$'s are measure independent.

(♣) Suppose that $\bigwedge_n [F_n] + X$ is null.

CLAIM: *There exist $K_n \subseteq {}^{[a_n,b_n)}2$ $(n \in \omega)$ with $|K_n| \leq c_n \cdot 2^{a_n}$ such that*

$$\forall\, x \in X\ \exists^\infty n\ x|[a_n, b_n) \in K_n.$$

Proof: Let $G \subseteq {}^\omega 2$ be an open set covering $\bigwedge_n [F_n] + X$ such that $\mu(G) < \prod_n \epsilon_n$ (as in Lemma 1, $\epsilon_n = 1 - 2^{-(n+1)}$). Let

$$K_n = \{\sigma|[a_n, b_n)\colon \sigma \in {}^{[a_n,a_{n+1})}2\ \&\ \exists\ \tau \in {}^{a_n}2\ F_n + \sigma \subseteq L_\tau\},$$

where

$$L_\tau = \{\sigma \in {}^{[a_n,a_{n+1})}2\colon \mu(G_{\tau\cup\sigma}) > \mu(G_\tau)/\epsilon_n\}.$$

We have

$$|K_n| \leq c_n \cdot 2^{a_n}.$$

Indeed, fix $\tau \in {}^{a_n}2$. By the Fubini theorem applied to $G_\tau \subseteq {}^{[a_n,a_{n+1})}2 \times {}^{[a_{n+1},\omega)}2$, $\mu(L_\tau) < \epsilon_n$. Let $\sigma_k \in {}^{[a_n,a_{n+1})}2$ $(k < k_\tau)$ be such that $F_n + \sigma_k \subseteq L_\tau$ and all $\sigma_k|[a_n, b_n)$ are distinct. Then

$$\bigcap_k {}^{[a_n,a_{n+1})}2 \setminus (F_n + \sigma_k) \supseteq {}^{[a_n,a_{n+1})}2 \setminus L_\tau.$$

So, by independence,

$$(1-2^{-n})^{k_\tau} \geq 1-\mu(L_\tau) > 1-\epsilon_n = 2^{-(n+1)},$$

hence $k_\tau \leq c_n$. The estimation for $|K_n|$ follows.

The rest of the proof of the claim is as in Lemma 1. ∎ (Claim)

Let now $K_n = \{\tau_n^i : i < c_n \cdot 2^{a_n}\}$. By the claim, for any $t \in {}^\omega 2$ such that

$$\bigcup_{n,i} \tau_n^i | [a_n^i, a_n^{i+1}) \subseteq t$$

we have

$$t + X \subseteq \bigvee_n [T_n].$$

Indeed, if $\tau_n^i \subseteq x \in X$, then $(t+x)|[a_n^i, a_n^{i+1}) \equiv 0$. ∎ (♣)

(♠) Suppose that $\bigvee_n [F_n] + X$ is null.

CLAIM: *There exist $K_n \subseteq {}^{[a_n,b_n)}2$ $(n \in \omega)$ with $|K_n| \leq c_n \cdot 2^{a_n}$ such that*

$$\forall x \in X \quad \forall^\infty n \quad x|[a_n, b_n) \in K_n.$$

Proof: Let G be an open set covering $\bigvee_n [F_n] + X$ such that $\mu(G) < 1$ and for every $\tau \in {}^{<\omega}2$ with $[\tau] \not\subseteq G$ we have $\mu([\tau] \setminus G) > 0$. For such τ let

$$K_{\tau,n} = \{\sigma|[a_n, b_n) : \sigma \in {}^{[a_n,a_{n+1})}2 \ \&\ [F_n + \sigma] \cap ([\tau] \setminus G) = \emptyset\}.$$

We have

$$\sum_n |K_{\tau,n}| \cdot 2^{-n} < \infty.$$

Indeed, let $k_n = |K_{\tau,n}|$ and choose $\sigma_n^k \in {}^{[a_n,a_{n+1})}2$ $(k < k_n)$ so that $\sigma_n^k|[a_n, b_n)$'s are distinct and give all $K_{\tau,n}$. Then

$$\bigcap_{n,k} {}^\omega 2 \setminus [F_n + \sigma_n^k] \supseteq [\tau] \setminus G.$$

So, by independence,

$$\prod_n (1-2^{-n})^{k_n} > 0.$$

It follows that

$$\sum_n k_n \cdot 2^{-n} < \infty.$$

For each τ as above choose now $n_\tau \in \omega$ so that

$$\sum_{\tau}\sum_{n\geq n_\tau} |K_{\tau,n}|\cdot 2^{-n} < \infty.$$

Let

$$K_n = \bigcup\{K_{\tau,n}\colon \tau \text{ is such that } n_\tau \leq n\}.$$

Then $\sum_n |K_n|\cdot 2^{-n} < \infty$, so

$$\forall^\infty n \quad |K_n| \leq 2^n \leq c_n\cdot 2^{a_n}.$$

We shall, without loss of generality, drop ∞ in this estimation.

Fix now $x \in X$. We shall show that

$$\forall^\infty n \quad x|[a_n, b_n) \in K_n.$$

We have

$$(\bigvee_n [F_n] + x) \cap ({}^\omega 2 \setminus G) = \emptyset.$$

By Baire's category theorem applied to ${}^\omega 2 \setminus G$, there is $m \in \omega$ and $\tau \in {}^{<\omega}2$ with $[\tau] \cap ({}^\omega 2 \setminus G) \neq \emptyset$ such that

$$\big(\bigcup_{n\geq m} [F_n + x|[a_n, a_{n+1})]\big) \cap ([\tau] \setminus G) = \emptyset.$$

Then for $n \geq \max(n_\tau, m)$ we have

$$x|[a_n, b_n) \in K_n. \qquad \blacksquare \text{ (Claim)}$$

Let now $K_n = \{\tau_n^i\colon i < c_n\cdot 2^{a_n}\}$. By the claim, for any $t \in {}^\omega 2$ such that

$$\bigcup_{n,i} \tau_n^i|[a_n^i, a_n^{i+1}) \subseteq t$$

we have

$$t + X \subseteq \bigwedge_n [T_n]. \qquad \blacksquare\ (\spadesuit)$$

$\blacksquare$

Bartoszyński (personal communication) noted that Shelah's [Sh] proof gives a null set $G \subseteq {}^\omega 2$ such that any $X \subseteq {}^\omega 2$ for which $G + X$ is null can be translated into G. Since the set G was in a natural way obtained as a union of two null sets, it seemed improbable that G itself could be small, where small is taken in the sense of Bartoszyński [B]. (A set $G \subseteq {}^\omega 2$ is small if there is a partition of ω into finite sets A_n $(n \in \omega)$ and there exist $S_n \subseteq {}^{A_n}2$ $(n \in \omega)$ such that $G \subseteq \bigvee_n [S_n]$ and $\sum_n \mu(S_n) < \infty$. Bartoszyński [B] showed that every null set is a union of two small sets and that there exist null sets that are not small.)

Using Proposition 1 we can find a set G which is small and has the above properties.

COROLLARY: *There exists a small set $G \subseteq {}^\omega 2$ such that any $X \subseteq {}^\omega 2$ for which $G + X$ is null can be translated into G.*

Proof: In the notation of Proposition 1 take $D = \emptyset$ and let $G = \bigwedge_n [F_n] \cup \bigvee_n [T_n]$ (or, $G = \bigvee_n [F_n] \cup \bigwedge_n [T_n]$). Then G is small. Indeed, let $A_n = [a_n, a_{n+1})$ and $S_n = F_n \cup T_n$ $(n \in \omega)$. Then $G \subseteq \bigvee_n [S_n]$ and $\forall n\ \mu(S_n) \leq 2^{-n+1}$. Now, if $G + X$ is null, then $\bigwedge_n [F_n] + X$ (resp. $\bigvee_n [F_n] + X$) is null, so, by (♣) (resp. (♠)), X can be translated into $\bigvee_n [T_n] \subseteq G$ (resp. $\bigwedge_n [T_n] \subseteq G$). ∎

Recently Andryszczak and Recław [AR] strengthened the (a) ⇒ (b) implication of the Galvin–Mycielski–Solovay characterization to: if $X \subseteq \mathbf{R}$ is strongly null then for every $\mathbf{G}_\delta$ set $G \subseteq \mathbf{R} \times \mathbf{R}$ all of whose vertical sections G_s $(s \in \mathbf{R})$ are dense, $\bigcap_{x \in X} G_x \neq \emptyset$. (This was also known to Galvin.)

The following proposition shows how the Andryszczak–Recław result can be obtained from the Galvin–Mycielski–Solovay characterization.

PROPOSITION 2: *Let $G \subseteq {}^\omega 2 \times {}^\omega 2$ be a $\mathbf{G}_\delta$ (resp. open) set with all vertical sections G_s $(s \in {}^\omega 2)$ dense. Then there is a dense $\mathbf{G}_\delta$ (resp. open) set $H \subseteq {}^\omega 2$ and a continuous function $f\colon {}^\omega 2 \to {}^\omega 2$ such that $\forall s\ f[H + s] \subseteq G_s$. In particular, if $X \subseteq {}^\omega 2$ and $t \in {}^\omega 2$ are such that $t + X \subseteq H$, then $f(t) \in \bigcap_{x \in X} G_x$.*

Proof: We do it for $\mathbf{G}_\delta$. Find increasing $\langle a_n : n \in \omega \rangle$, $\langle b_n : n \in \omega \rangle \in {}^\omega\omega$, $a_0 = 0$, and $\phi(\tau) \in {}^{[a_n, a_{n+1})}2$ $(\tau \in {}^{b_n}2)$ such that

$$\bigcap_m \bigcup_{n>m} \bigcup_{\tau \in {}^{b_n}2} [\tau] \times [\phi(\tau)] \subseteq G.$$

Next choose $\tau_n \in {}^{b_n}2$ $(n \in \omega)$ so that

$$H = \bigcap_m \bigcup_{n>m} [\tau_n]$$

is dense and define f by

$$f(t) = \bigcup_n \phi(\tau_n + t|b_n).$$

The function f is clearly continuous. We shall prove that $f[H+s] \subseteq G_s$. To see this let $t \in [\tau_n] + s$. Then $s \in [\tau_n] + t = [\tau_n + t|b_n]$ and $f(t) \in [\phi(\tau_n + t|b_n)]$. So, if $\exists^\infty n\ t \in [\tau_n] + s$, then

$$\exists^\infty n\ \exists \tau \in {}^{b_n}2\ \langle s, f(t)\rangle \in [\tau] \times [\phi(\tau)],$$

implying $\langle s, f(t)\rangle \in G$. ∎

Notes: (0) Lemma 2 is just the easy (c) $\Rightarrow$ (a) implication of the Galvin–Mycielski–Solovay theorem. A combination of Lemmas 1 and 3 gives the hard implication (a) $\Rightarrow$ (b). A direct proof might be as follows. Given a comeager set H, find an increasing sequence $\langle a_n \colon n \in \omega\rangle \in {}^\omega\omega$ and sequences $\sigma_n \in {}^{[a_n, a_{n+1})}2$ ($n \in \omega$) such that $\bigvee_n [\sigma_n] \subseteq H$. If X is strongly null, there exist $\tau_n \in {}^{[a_n, a_{n+1})}2$ ($n \in \omega$) such that $X \subseteq \bigvee_n [\tau_n]$. Let $t = \bigcup_n \sigma_n + \tau_n$. Then

$$t + X \subseteq \bigvee_n [\sigma_n + \sigma_n + \tau_n] = \bigvee_n [\tau_n] \subseteq H.$$

(1) Note that f in Proposition 2 can be chosen to be one-to-one (choose ϕ to be one-to-one). Also, the proposition can be reformulated as follows. If $G \subseteq {}^\omega 2 \times {}^\omega 2$ is a $\mathbf{G}_\delta$ (resp. open) set with all vertical sections G_s ($s \in {}^\omega 2$) dense, then there is a dense $\mathbf{G}_\delta$ (resp. open) set $H \subseteq {}^\omega 2$ and a continuous function $f\colon {}^\omega 2 \to {}^\omega 2$ such that $\forall t\ H + t \subseteq G^{f(t)}$ (the upper-script means horizontal section). Indeed, instead of $\forall s\ f[H+s] \subseteq G_s$, as in the proposition, we can write $f^*[H^*] \subseteq G$, where $H^* = \bigcup_{s \in {}^\omega 2} \{s\} \times (H+s)$ and $f^*\colon {}^\omega 2 \times {}^\omega 2 \to {}^\omega 2 \times {}^\omega 2$, $f^*(\langle s,t\rangle) = \langle s, f(t)\rangle$. Then $\forall t\ (H^*)^t \subseteq G^{f(t)}$ & $(H^*)^t = H + t$.

The proposition remains true for $\mathbf{R}/\mathbf{Z}$ (in fact, any compact Polish group). For $\mathbf{R}$ (in general, locally compact) it is true if we drop the 'open' part. To see that we have to do this: Let $G \subseteq \mathbf{R} \times \mathbf{R}$ be an open set with all vertical sections G_s ($s \in \mathbf{R}$) dense and such that $\forall t \in \mathbf{R}\ \forall n \in \omega\ G^t \cap [n, \infty)$ contains no interval of size 2^{-n} (e.g., $G = \{\langle s,t\rangle \colon \forall n \in \omega\ (s \in [n, n+1] \Rightarrow \forall k \in \mathbf{Z}\ t \neq s + k \cdot 2^{-n-1})\}$). Then, if $H \cap [0,\infty)$ contains an interval of size 2^{-n}, then $(H+n) \cap [n, \infty)$ can't be covered by any G^t.

(2) Using the Andryszczak–Recław result it is not hard to see that the set $\mathbf{R} \setminus (D+X)$ in the Galvin–Mycielski–Solovay characterization is fairly thick. E.g.,

if X is strongly null, G an uncountable $\mathbf{G}_\delta$ and D such that $G\cap(D+x)$ is meager in G for all $x \in \mathbf{R}$, then $G \setminus (D+X)$ contains a nonempty perfect set (see [P]). Note, however, that we can't claim that $D + X$ is meager. The continuum hypothesis implies that there exists a nonmeager strongly null set (such is a Lusin set, see [Mi]).

(3) It is not hard to strenghten the $\Rightarrow$ implication of our theorem to: if $X \subseteq \mathbf{R}$ is strongly null then for every closed $F \subseteq \mathbf{R} \times \mathbf{R}$ with all vertical sections F_s ($s \in \mathbf{R}$) null, $\bigcup_{x\in X} F_x$ is null (see [P]).

A nonmeager strongly null set X shows again that in the $\Rightarrow$ implication of our theorem we can't require that $F + X$ is coverable by a null $\mathbf{F}_\sigma$ set (null $\mathbf{F}_\sigma$ sets are meager). It also shows that we can't drop the requirement that F is closed. (By a theorem of Steinhaus, if A is nonmeager and B comeager then $A+B = \mathbf{R}$, so, if X is nonmeager strongly null and F is comeager null, then $F + X = \mathbf{R}$.)

(4) Galvin [G] shows that if $X \subseteq \mathbf{R}$ is such that for any dense $\mathbf{G}_\delta$ set $G \subseteq \mathbf{R}$ there exist $a \neq 0$ and b with $a \cdot X + b \subseteq G$, then X is strongly null. Passing to the complement of G we get that if $X \subseteq \mathbf{R}$ is such that for any meager $D \subseteq \mathbf{R}$ there exists $a \neq 0$ with $D + a \cdot X \neq \mathbf{R}$, then X is strongly null.

The $\Leftarrow$ implication of our theorem can be strengthened in a similar way. Namely, if $X \subseteq \mathbf{R}$ is such that for any closed null set $F \subseteq \mathbf{R}$ there exists $a \neq 0$ with $F + a \cdot X$ null, then X is strongly null. Indeed, by Lemma 1, if for every closed null set $F \subseteq \mathbf{R}$ there is a with $F + a \cdot X$ null, then for every meager set $D \subseteq \mathbf{R}$ there is a with $D + a \cdot X \neq \mathbf{R}$.

(5) Recall that a set $X \subseteq \mathbf{R}$ is strongly meager iff for every null set $G \subseteq \mathbf{R}$, $G + X \neq \mathbf{R}$ (see [Mi]). One may think about the following dual theorem: a set $X \subseteq \mathbf{R}$ is strongly meager iff for every closed null set $F \subseteq \mathbf{R}$, $F + X$ is meager.

The $\Rightarrow$ implication is an old unpublished result of Recław. A short proof might be as follows. Let F, a_n's and F_n's be as in the proof of Lemma 3. Then $\bigvee_n[F_n]$ is null and $F \subseteq \bigwedge_n[F_n]$. Since X is strongly meager, $\bigvee_n[F_n] + X \neq {}^\omega 2$. By Lemma 4, $\bigwedge_n[F_n] + X$ is meager.

The $\Leftarrow$ implication is false: Suppose that in V the union of any $\aleph_1$ meager sets is meager. Let c be a Cohen real over V. Then in $V[c]$ we have: (♣) there is a null set G such that for every uncountable $X \subseteq \mathbf{R}^V$, $G + X = \mathbf{R}$ ([Ca]); (♠) the union of any $\aleph_1$ meager sets is meager. So, in $V[c]$ ([CP]), if $X \subseteq \mathbf{R}^V$ has size $\aleph_1$, then X is not strongly meager (by (♣)) and $F + X$ is meager for all meager F (by (♠)).

Another attempt at a dual theorem can be: a set $X \subseteq \mathbf{R}$ is strongly meager iff for every nowheredense set $F \subseteq \mathbf{R}$, $F + X$ is meager. This is false in both directions. For $\Leftarrow$ we argue as before. For $\Rightarrow$, if X is a nonnull strongly meager set (e.g., a Sierpiński set, see [P]) and F is a co-null meager set, then, by a theorem of Steinhaus, $F + X = \mathbf{R}$.

QUESTION: *Can we replace in our characterization* $\mathbf{R}$ *by* $(\mathbf{R}/\mathbf{Z})^\omega$*?*

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